RECURRENCE RELATIONS FOR THE MOMENTS OF ORDER STATISTICS FOR DISCRETE DISTRIBUTIONS

By

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1. Introduction

Some recurrence relations between the mixed moments and between the expected values of functions of order statistics have been obtained by Joshi [2] and Krishnaih and Rizvi [4] respectively, assuming the sample observations from the absolutely continuous population. The aim of this paper is to generalize those relations for discrete population. The author feels that the relations also hold even for mixed and exchangeable variates.

2. Assumptions and Notations

Let X be an integral-valued random variable with probability mass function $P\ (X=x)=p(x)$ for $x=-, 2, -1, 0, 1, 2, \ldots$ and cumulative distribution function (cdf) $P(x)=\sum_{i\leqslant x} p(i)$. Let $X_{1:n}\leqslant X_{2:n}$ $\ldots\leqslant X_{n:n}$ be the ordered sample from a discrete population. The first moment $E(X_{r:n})$ and the mixed moment $E(X_{r:n}\ X_{s:n})$ are denoted by $\mu_{r:n}\ (1\leqslant r\leqslant n)$ and $\mu_{rs:n}\ \ (1\leqslant r<s\leqslant n)$ respectively and the existence of these moments is assumed.

3. Recurrence Relations

Let

$$W_1=\{\ (x, y)\ :\ -\infty<x\leqslant<\infty\ \}$$
$$W_2=\{\ (x, y)\ :\ -\infty<x\leqslant y<\infty\ \}$$
$$R=W_1\cup W_2$$

$$B\ (p, q, r)=\frac{\Gamma(p)\ \Gamma(q)\ \Gamma(r)}{\Gamma(p+q+r)}\ ,\ p, q, r>0$$

and following by Khatri (1962)

$$P(X_{r:n}=x)=\frac{1}{B(r,n-r+1)}\int_{P(x-1)}^{P(x)} w^{r-1}(1-w)^{n-r}\,dw$$

$$P(X_{r:n}=x,\ X_{s:n}=y)\frac{1}{B(r,s-r,n-s+1)}\iint w^{r-1}(v-w)^{s-r-1}(1-v)^{n-s}\,dv\,dw$$

where the integration is to be carried out over the region : $v \geqslant w$, $P(x) \geqslant w \geqslant P(x-1)$ and $P(y) \geqslant v \geqslant P(y-1)$

Then for $1 \leqslant r \leqslant n$

$$(3.1)\quad \mu_{r:n}=\frac{1}{B(r,n-r+1)}\sum_{-\infty}^{\infty}\int_{P(x-1)}^{P(x)} w^{r-1}(1-w)^{n-r}\,dw$$

and for $1 \leqslant r < s \leqslant n$

$$(3.2)\quad \mu_{rs:n}=\frac{1}{B(r,s-r,n-s+1)}\sum_{W_1} x\,y \iint w^{r-1}(v-w)^{s-r-1}(1-v)^{n-s}\,dv\,dw$$

$$=\frac{1}{B(r,s-r,n-s+1)}\sum_{W_2} x\,y \iint v^{r-1}(w-v)^{s-r-1}(1-w)^{n-s}\,dv\,dw$$

Theorem 1. For $1 \leqslant r \leqslant n-1$

$$(3.3)\quad B(1,n-r,r)\ \mu_{rn:n}+\sum_{i=0}^{r-1}(-1)^{n-i}\binom{r-1}{i}B(1,n-r,r-i)$$

$$\mu_{1,\,n-r+1:n-i}=\sum_{i=1}^{n-r}(-1)^{n-r-i}\binom{n-r-i}{i-1}\frac{\mu_{i:i}}{i}\ \frac{\mu_{n-i:\,n-i}}{n-i}$$

Proof.

Consider

$$S=\sum_{R} x\,y\iint w^{r-1}(v-w)^{n-r-1}\,dv\,dw$$

$$=\sum_{R} x\,y\iint w^{r-1}$$

$$\sum_{i=1}^{n-r}\binom{n-r-1}{i-1}(-1)^{n-r-i}\,w^{n-r-i}\,v^{i-1}\,dv\,dw$$

$$= \sum_{i=1}^{n-r} (-1)^{n-r-i} \binom{n-r-1}{i-1} \frac{\mu_{i:i}}{i} \quad \frac{\mu_{n-i:n-i}}{n-i}$$

Again

$$S = \sum_{W_1} x\, y \iint w^{r-1} (v-w)^{n-r-1}\, dv\, dw$$

$$+ \sum_{W_2} x\, y \iint v^{r-1} \; (w-v)^{n-r-1}\, dv\; dw$$

$$= \mu_{rn:n}\; B\,(r,\, n-r+1)$$

$$+ \sum_{W_2} x\, y \iint \quad v^{r-1} \; (w-v)^{n-r-1}\, dv\; dw\,.$$

Now

$$\sum_{W_2} x\, y \iint \quad v^{r-1}\, (w-v)^{n-r-1}\, dv\; dw$$

$$= \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^{n-i}$$

$$\sum_{W_2}^{r-1} \iint \quad (v-w)^{n-r-1}\, (1-v)^{r-i-1}\, dv\; dw$$

$$= \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^{n-i}\, B\,(1,\, n-r,\, n-r)$$

$$\mu_{1,\, n-r+1\,:\, n-i}$$

Hence the theorem.

Corollary. If the parent distribution is symmetrical about zero, then

(3.4) $$\{\, 1+(-1)^{n}\, B\,(1,\, n-r,\, r)\,\}\; \mu_{1,\, n-r+1\,:\, n}$$

$$= \sum_{i=1}^{r-1} \quad (-1)^{n-r+1} \binom{r-1}{i} B\,(1,\, n-r,\, r-i)$$

$$\mu_{1,\, n-r+1:n-i}$$

$$+ \sum_{i=2}^{n-r} \; (-1)^{n-r-i} \binom{n-r-1}{i-1} \frac{\mu^{i}\,:\,i}{i} \; \frac{\mu_{n-i\,:\,n-t}}{n-i}$$

Lemma. For symmetrical distribution zero, $\mu_{1:1}=0$ and $\mu_{r,n:n}=\mu_{1,n-r+1:n}$.

Proof of the Lemma. Putting $r=n=1$ in (3.1) and using the fact $p(x)=p(-x)$, $P(x)=1-P(-x-1)$, $P(x-1)=1-P(-x)$ for discrete symmetrical distribution about zero, we get $\mu_{1:1}=0$. Similiarly $\mu_{rn:n}=\mu_{1,n-r+1:n}$ can be proved.

Using the lemma in (3.2), (3.4) is obtained. It is concluded that for symmetrical population, the mixed moments $\mu_{1s:n}$ can only be obtained for even values of n when the first and mixed moments of order statistics for a sample less than n are available. For odd values of n, the relations involving the first and mixed moments in samples of size $n-1$ or less is obtained. For particular values of r we can draw the same conclusion as given by Joshi (1971).

Theorem 2. For $1 \leqslant r \leqslant m \leqslant n$

$$(3.5)\quad E(h(X_{r:m}))=\binom{m}{r}\sum_{s=0}^{i}(-1)^s\left(\frac{r}{r-i}\right)\left[\binom{i}{s}\Big/\binom{m-i+s}{r-i}\right]0\leqslant i\leqslant r-1$$
$$\cdot\, E(h(X_{r-i:m-i+s}))$$

and also

$$(3.6)\quad E(h(X_{r:m}))=\binom{m}{r}\sum_{s=0}^{j}(-1)^s\left(\frac{r}{r+s}\right)\left[\binom{j}{s}\Big/\binom{m-j+s}{r+s}\right]\cdot E(h(X_{r+s:m-j+s}))\ 0\leqslant j\leqslant m-r$$

where $h(.)$ is an arbitrary specified function such that $E[h(X)]$ exists.

Proof. For $1 \leqslant r \leqslant m \leqslant n$ and $0 \leqslant i \leqslant r-1$, we have

$$E[h(X_{r:m})]=\sum_{-\infty}^{\infty}h(x)\binom{m}{r}\int_{P(x-1)}^{P(x)}w^{r-1}(1-w)^{m-r}\,dw$$
$$=r\binom{m}{r}\sum_{-\infty}^{\infty}h(x)\int_{P(x-1)}^{P(x)}$$
$$\sum_{s=0}^{i}(-1)^s\binom{i}{s}w^{r-i+1}(1-w)^{m-r+s}\,dw$$

$$= \binom{m}{r} \sum_{s=0}^{i} (-)^s \left[\binom{i}{s} / \binom{m-i+s}{r-i} \right] \frac{r}{r-i} \Big) \; E\left(h\left(X_{r-i:m-i+s}\right)\right)$$

Also for $1 \leqslant r \leqslant m$ and $0 \leqslant j \leqslant m-r$ we have

$$E\left(h\left(X_{r:m}\right)\right) = \sum_{-\infty}^{\infty} h(x)\, r \binom{m}{r} \int_{P(x-1)}^{P(x)} w^{r-1} \sum_{s=0}^{j} (-1)^s \binom{j}{s} w^s (1-w)^{m-r-j}\, dw$$

$$= \binom{m}{r} \sum_{s=0}^{j} (-1)^s \left[\binom{j}{s} / \binom{m-j+s}{r+s} \right] \left(\frac{r}{r+s} \right) E\left(h\left(X_{r+s:m-j+s}\right)\right)$$

On the choice of $h(x)$, the relations between the moments, between the moment generating functions, between the characteristic functions and between the *pdf's* of order statistics from discrete population can be obtained.

This is well known (David 1970, p. 38) that

$$E(X_{r:n}) = \sum_{i=r}^{n} \binom{i-1}{r-1} \binom{n}{i} (-1)^{i-r} E(X_{i:i})$$

showing that the expected value of $X_{r:n}$ is expressible in terms of the expected value of the maximum in samples of sizes $r, r+1, \ldots, n$. If we choose r, a function of n, say $r(n)$ such that $r(n)/n \to p$ $(0 < p < 1)$, a constant as $n \to \infty$ representing $X_{r:n}$, a sample quantile of order p and thereby the expected value of a sample quantile may be computed.

In a similar manner the expected value of sample quantile can be expressed in terms of the expected value of the minimum in samples of sizes $n-r+1, n-r+2, \ldots, n$ by using the recurrence relation

$$E(X_{r:n}) = \sum_{i=n-r+1}^{n} \binom{i-1}{n-r} \binom{n}{i} (-1)^{i-n+r-1} E(X_{1:i})$$

The expected value of a sample quantile of a certain order can also be expressed in terms of the sample quantiles of higher order in samples of $n-k$, $n-k+1$, .. , n by the relation given below

$$E(X_{r:n}) = \sum_{j=0}^{k} (-1)^j \left(\frac{r}{j+r}\right)\left[\binom{n}{r} / \binom{n-k+j}{j+r}\right] E(X_{j+r\,;\,n-k+j})$$

Hence $(j+r)/(n-k+j)$ does not approach to 0 or 1 as $n \to \infty$ but $(j+r)/(n-k+j)$ converges to $\delta\ (0 < \delta < 1)$, a constant.

Let $h(\ldots)$ be a specified function such that $E(h(x, Y))$ exists, then for $1 \leqslant r < s \leqslant n$

$$E(h(X_{r:n}, X_{s;n}))$$

$$= \frac{1}{B(r, s-r, n-s+1)} \iint h(w, v)\, w^{r-1} (v-w)^{s-r-1} (1-v)^{n-s}\, dw\, dv$$

where the integration is to be carried out over the region : $v \geqslant w$. $P(x) \geqslant w \geqslant P(x-1)$, $P(y) \geqslant v \geqslant P(y-1)$

Using the identity

$$w^{r-1}(v-w)^{s-r-1}(1-v)^{n-s} = w^{r-1}(v-w)^{s-r-1}(1-v)^{n-s-1}$$
$$-w^{r-1}(v-w)^{s-r}(1-v)^{n-s-1} - w^{r}(v-w)^{s-r-1}(1-v)^{n-s-1}$$

we have

$$r\,E(h(X_{r+1:n}, X_{s+1:n})) = n\,E(h(X_{r:n-1}, X_{s:n-1}))$$
$$-(n-s)\,E(h(X_{r:n}, X_{s;n})) - (s-r)\,E(h(X_{r:n}, X_{s+1:n}))$$

If we let $h(X_{r:n}, X_{s:n}) = 1$, we have a recurrence relation between the pmf of order statistics. Further if we let $h(X_{r:n}, X_{s:n}) = X_{r:n} X_{s:n}$, the recurrence relation for the moments of order statistics is obtained.

References

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[4] Krishnaih, P.R. and Rizvi, M. (1966) : A note on recurrence relations between the expected values of functions of order statistics. *Ann. Math. Statist.* 37, 733-34.